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Correlation functions of dense polymers and $c = -2$ conformal field theory

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Abstract. 'Antisymmetric' correlation functions of the model of dense lattice polymers are proved to be given by the generalized Kirchhoff theorem. In the continuum limit they coincide with the correlation functions of the free complex Grassmann field that corresponds to the non-unitary conformal field theory (CFT) with $c = -2$. Explicit expressions for the correlation functions are found. These do not obey standard Wick rules due to the presence of zero mode. Nevertheless, the complex Grassmann field θ can be considered as primary with conformal weight $(h_\theta, \bar{h}_\theta) = (0, 0)$. It is shown that the natural space of states wherein the operators of the theory act is the Krein space with indefinite metric.

1. Introduction

The model of lattice polymers has a long history, dating back more then a century, when Kirchhoff proved a beautiful theorem that the number of spanning trees (branching or dense polymers) on the lattice of N sites is given by the principal minors of the $N \times N$ matrix of the discrete Laplacian [1, 2]. Later it was realized that the statistics of polymers is closely related to the statistics of spin models. In particular, Fortuin and Kasteleyn observed [3, 4] that the partition function Z_N of the q-component Potts model can be represented as a dichromatic polynomial that continuously depends on q. Although the partition function of the model vanishes in the formal limit $q \to 0$ owing to zero mode of the discrete Laplacian, its derivative with respect to q does not and gives the partition function of dense lattice polymers (spanning trees). At about the same time de Gennes [5] explained how the partition function of dilute polymers can be obtained from the partition function of the $O(n)$-model in the formal limit $n \to 0$. The nature of the phase transition from the high-temperature (dilute) phase to the low-temperature (dense) phase has been the subject of many investigations [6–11]. Using a series of model transformations Nienhuis [12] has shown how a particular $O(n)$ model on the hexagonal lattice can be mapped onto a coulombic gas. The properties of the critical point can be deduced from this mapping and the exponents obtained in this way are in good agreement with numerical estimates. Then, Parisi and Sourlas [13] explained how the limit $n \to 0$ can be avoided by considering the supersymmetric generalization of the Hamiltonian of the $O(n)$ model. They argued that the phase transition from the dilute to the dense phase corresponds to the breakdown of supersymmetry. Saleur [14] suggested using $N = 2$ superconformal field theory to describe the properties of the dilute polymers. In this approach the dense phase has been described by the ghost (ξ, η)-system with central charge $c = -2$.

In spite of all these efforts, the non-unitary conformal field theories (CFTs) that describe the continuum limit of lattice polymers so far have not been fully understood. It is the purpose of this paper to clarify some of the mathematical structures behind the non-unitary CFTs. Namely, the following will be shown.

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(i) Although the partition function of the q-component Potts model vanishes in the formal limit $q \to 0$, some 'antisymmetric' 2γ-point correlation functions survive. These correlation functions are given exactly by the minors of rank $(N-\gamma)$ of the Laplacian matrix. These can be rewritten in terms of the integrals over anti-commuting variables and in the continuum limit coincide with the correlation functions of the free complex Grassmann field. Explicit expressions for the correlation functions are found which do not obey standard Wick rules since the vacuum vector of the field theory has zero norm.

(ii) The natural space of states wherein the operators of the theory act is the Krein space with indefinite metric [15]. The Krein space $\mathcal{H}$ is defined as a Hilbert space with definite metric $(x|y)$, and additional involutive unitary operator of canonical symmetry J. An explicit expression for the operator is given in terms of zero modes of the Grassmann field.

(iii) The basic property of the $c = -2$ CFT is that the correlation functions of the primary fields with positive conformal weights are non-zero and satisfy the standard differential equations of CFT only when calculated in the definite metric. In contrast, the correlation functions of the primary fields with non-positive conformal weights should be defined in the indefinite metric. Alternatively, the zero modes can be supressed by the Dirichlet operator that imposes Dirichlet boundary conditions.

2. Dense lattice polymers

Let the lattice $\mathcal{L}$ have N sites labelled $1, 2, \ldots, N$. With each site i we associate a spin variable σ_i which can take q values, say $1, 2, \ldots, q$. Then the average of any operator $\mathcal{A}(\sigma)$ in the q-component Potts model we define as (without a normalization factor!)

$$\langle \mathcal{A}(\sigma) \rangle = \sum_{\sigma} \mathcal{A}(\sigma) \exp\left\{ \beta J \sum_{(ij)} \delta(\sigma_i, \sigma_j) \right\}. \tag{1}$$

Here σ-summation is over all spins $\sigma_1, \ldots, \sigma_N$; the second summation is over all edges of the lattice.

Setting $v = \exp(\beta J) - 1$, then the partition function can be rewritten as

$$Z_N = \langle 1 \rangle = \sum_{\sigma} \prod_{(ij)} (1 + v\delta(\sigma_i, \sigma_j)). \tag{2}$$

Let E be the number of edges of the lattice $\mathcal{L}$. Then the summand in equation (2) is a product of E factors. Each factor is the sum of two terms: 1 and $v\delta(\sigma_i, \sigma_j)$, so the product can be expanded as the sum of 2^E terms.

Each of these 2^E terms can be associated with a bond-graph on the lattice $\mathcal{L}$. To do this, note that the term is the product of E factors, one for each edge. The factor for edge (ij) is either 1 or $v\delta(\sigma_i, \sigma_j)$: if it is the former, leave the edge empty, if the latter, place a bond on the edge. Do this for all edges (ij). We then have a one-to-one correspondence between bond-graphs on $\mathcal{L}$ and terms in the expansion of the product in equation (2).

Considering a typical bond-graph $\mathcal{G}$, containing N sites, L bonds, γ connected components and ω internal cycles. These are not independent, but must satisfy Euler's relation

$$L + \gamma = N + \omega. \tag{3}$$

Then the corresponding term in the expansion contains a factor v^L and the effect of delta functions is that all sites within a component must have the same spin σ. Summing over all independent spins and over all bond-graphs $\mathcal{G}$ that can be drawn on $\mathcal{L}$ we obtain [3, 4]

$$Z_N = \sum_{\mathcal{G}} q^{\gamma} v^L. \tag{4}$$

Note that here q need not be an integer. We can formally allow it to be any real number and, in particular, to consider the formal limit $q \to 0$.

Since we are going to deal with not only one- but arbitrary γ-component spanning trees, we have to treat the limit in a different way from [3]. At first we consider the limit $\lambda, q, v \to 0$ while $\kappa = q/\lambda$ and $x = v/\lambda$ remain finite. As a result we obtain the partition function of spanning trees with arbitrary number of components (lattice forests)

$$\tilde{Z}_N = \lim_{\lambda\to 0} \lambda^{-N} Z_N = \lim_{\lambda\to 0} \sum_{\mathcal{G}} \kappa^{\gamma} \lambda^{\omega} x^L = \sum_{\mathcal{T}} \kappa^{\gamma} x^L. \tag{5}$$

Here the last summation is over all bond-graphs $\mathcal{T}$ that have no internal cycles, i.e. $\omega = 0$. Such graphs are usually called spanning trees. The number of bonds L of the spanning tree is related to the number of its components γ as $L = N - \gamma$. Hence, the partition function can be rewritten as

$$\tilde{Z}_N = \sum_{\gamma} \kappa^{\gamma} \sum_{\mathcal{T}_\gamma} x^L = \sum_{\gamma} \mathcal{N}_\gamma \kappa^{\gamma} x^{N-\gamma} \tag{6}$$

where $\mathcal{T}_\gamma$ denotes the set of different γ-component spanning trees and $\mathcal{N}_\gamma$ is their total number. To simplify further notations we take $x \equiv 1$ without loss of generality.

The second limit $\kappa \to 0$ leads to the critical point of the polymer model. Since $\gamma \geqslant 1$ the partition function in (6) obviously tends to zero in this limit. Nevertheless, the correlation functions do not necessarily vanish. Indeed, repeating all the steps leading to equation (6) one can prove that the following correlation functions survive in the limit $\kappa \to 0$

$$\lim_{\kappa,\lambda\to 0} \langle 1 \rangle = 0 \tag{7a}$$

$$\lim_{\kappa,\lambda\to 0} \left\langle \delta_{i_1 i_2} \right\rangle = \mathcal{N}_{(i_1 i_2)} = \text{constant} \tag{7b}$$

$$\lim_{\kappa,\lambda\to 0} \left\langle \begin{vmatrix} \delta_{i_1 i_3} & \delta_{i_1 i_4} \\ \delta_{i_2 i_3} & \delta_{i_2 i_4} \end{vmatrix} \right\rangle = \mathcal{N}_{(i_1 i_3)(i_2 i_4)} - \mathcal{N}_{(i_1 i_4)(i_2 i_3)} \tag{7c}$$

$$\cdots$$

Here $\delta_{i_1 i_2} = (v/q)\delta(\sigma_{i_1}, \sigma_{i_2})$; $\mathcal{N}_{(i_1 i_2)} = \mathcal{N}$ denotes the number of one-component spanning trees with both the sites i_1 and i_2 belonging to the same component (this number, obviously, does not depend on the position of the sites); $\mathcal{N}_{(i_1 i_3)(i_2 i_4)}$ is the number of two-component spanning trees with sites i_1, i_3 belonging to one component and sites i_2, i_4 to the other; etc.

The antisymmetric combination of δ in each 2γ-point correlation function is designed to guard against any contribution of spanning trees with the number of components less then γ (otherwise this would be divergent). So, only γ-component spanning trees contribute to the 2γ-point correlation function in the limit $\kappa \to 0$.

The importance of these 'antisymmetric' correlation functions is justified by the following result.

Theorem 1 (Generalized Kirchhoff Theorem). *Given a lattice $\mathcal{L}$ with N sites labelled $1, 2, ..., N$, the $N \times N$ matrix of discrete Laplacian Δ_{ij} has the elements: Δ_{ii} = number of edges incident to i, Δ_{ij} = −number of edges with end points i and j. The minor $\Delta^{(i_1)(i_2)}$ of rank $(N-1)$ is obtained from the matrix Δ by deleting i_1th column and i_2th row; similarly, the minor $\Delta^{(i_1 i_2)(i_3 i_4)}$ of rank $(N-2)$ is obtained by deleting columns $i_1 < i_2$ and rows $i_3 < i_4$; etc. Then the determinants*

$$\det \Delta = 0 \tag{8a}$$

$$\det \Delta^{(i_1)(i_2)} = \mathcal{N}_{(i_1 i_2)} = \text{constant} \tag{8b}$$

$$\det \Delta^{(i_1 i_2)(i_3 i_4)} = \mathcal{N}_{(i_1 i_3)(i_2 i_4)} - \mathcal{N}_{(i_1 i_4)(i_2 i_3)} \tag{8c}$$

$$\cdots$$

coincide with the 'antisymmetric' correlation functions of equation (7).

The standard proof of the original version of the theorem (first two lines of the sequence (8a, b)) can be found in [1]. Priezzhev [2] proposed an alternative proof of the original version in the spirit of the combinatorial solution of the Ising model. His method can easily be generalized to prove all other lines of equation (8).

There is a simple relation between the four-point correlation function (7c) and the Green function of the discrete Laplacian operator. The most straightforward way to understand this is follows. Let us replace every bond of the lattice $\mathcal{L}$ by a unit resistor and consider a current $I = 1$ entering this lattice of resistors at a site i_1 and leaving it at a site i_2. Then, the voltage difference between sites i_3 and i_4 on the lattice is proportional to the four-point function (7c)

$$G(i_1, i_2|i_3, i_4) = \frac{\mathcal{N}_{(i_1i_3)(i_2i_4)} - \mathcal{N}_{(i_1i_4)(i_2i_3)}}{\mathcal{N}}. \tag{9}$$

This statement is a simple generalization of a similar graphical representation for the Green function of the Laplacian operator with Dirichlet boundary conditions at the site i_0 [16]. Indeed, let us consider again the lattice of unit resistors grounded at the site i_0. This means that the voltage at this site is always maintained equal to zero. If a current $I = 1$ enters the lattice at a site i_1 (and leaves it at the grounded site i_0) then the voltage at a site i_3 is given by the Green function which has the following graphical representation

$$G_0(i_1|i_3) = \frac{\mathcal{N}_{(i_1i_3)(i_0)}}{\mathcal{N}} \tag{10}$$

where $\mathcal{N}_{(i_1i_3)(i_0)}$ is the number of two-component spanning trees with the sites i_1 and i_3 belonging to one component and the site i_0 to the other. This formula follows immediately from equation (9) if we consider the limit $i_2, i_4 \to i_0$ and take into account the fact that $\mathcal{N}_{(i_1i_0)(i_3i_0)} \equiv 0$.

3. Free complex Grassmann field

Using a matrix representation we can reinterpret the partition function of lattice polymers as being the partition function of some artificial statistical system. To this end we define at each site i of the lattice $\mathcal{L}$ the pair of anti-commuting variables θ_i and θ_i^* (its complex conjugate). Then, using the Berezin definition of the integral over anti-commuting variables [20] we can rewrite the determinant of the matrix Δ as

$$\begin{aligned}\det \Delta &= \int \mathrm{d}\theta_1^* \dots \mathrm{d}\theta_N \exp \sum_{(ij)} \theta_i^* \Delta^{ij} \theta_j \\ &= \int \mathrm{d}\theta_1^* \dots \mathrm{d}\theta_N \exp \sum_{(ij)} (\theta_i^* - \theta_j^*)(\theta_i - \theta_j).\end{aligned} \tag{11}$$

In the continuous limit this partition function defines field theory with the action

$$\mathcal{S}(\theta) = \frac{1}{4\pi} \int \partial_\mu \theta^* \partial^\mu \theta \, \mathrm{d}^2 \boldsymbol{r}. \tag{12}$$

The average of any operator $\mathcal{A}[\theta]$ we define as

$$\langle \mathcal{A}(\theta) \rangle = \int (\mathrm{d}\theta^* \, \mathrm{d}\theta] \mathcal{A}[\theta] \exp -\mathcal{S}(\theta). \tag{13}$$

The accurate consideration of the continuous limit of the determinants of (8) leads to the following explicit expressions for the correlation functions of the free complex Grassmann field

$$\langle 1 \rangle = 0 \tag{14a}$$

$$\langle \theta_1^* \theta_2 \rangle = 1 \tag{14b}$$

$$\langle\theta_1^*\theta_2^*\theta_3\theta_4\rangle = \ln(\eta_{34}^{12}) \tag{14c}$$

$$\langle\theta_1^*\theta_2^*\theta_3^*\theta_4\theta_5\theta_6\rangle = \begin{vmatrix} \ln(\eta_{45}^{12}) & \ln(\eta_{56}^{12}) \\ \ln(\eta_{45}^{23}) & \ln(\eta_{56}^{23}) \end{vmatrix} \tag{14d}$$

$\ldots.$

Note, that the field θ is scalar and its correlation functions depend only on the projectively invariant cross-ratios

$$\eta_{34}^{12} = \left(\frac{r_{14}r_{23}}{r_{13}r_{24}}\right)^2 \ldots. \tag{15}$$

Here $\theta_1 \equiv \theta(\boldsymbol{r}_1)$; $r_{12} \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. These correlation functions are asymptotics of the 'antisymmetric' correlation functions of (7) in the continuum limit. The first two lines of the sequence from equations (14*a*, *b*) have already appeared in [17].

The correlation functions of equation (14) look very unusual from the point of view of standard field theory. First, the correlation functions of the free Grassmann field do not obey standard Wick rules due to the presence of zero mode. Nevertheless, the Grassmann field can be considered as the primary conformal field with the weight $(h_\theta, \bar{h}_\theta) = (0, 0)$ (see [18] for definitions).

To prove this, one has to consider the stress-energy tensor

$$T(z) =: \partial\theta^*\partial\theta := \lim_{w\to z}\left\{\partial\theta^*(z)\partial\theta(w) + \frac{1}{(z-w)^2}\right\} \tag{16}$$

and verify by direct calculation of the contour integral that this tensor is indeed the generator of conformal transformations in the sense that for any correlation function $\langle X\rangle = \langle\theta_1^*\ldots\theta_{2N}\rangle$ from the sequence of (14) its transformation law is given by

$$\delta_\epsilon\langle X\rangle = \oint_C \mathrm{d}z\,\epsilon(z)\langle T(z)X\rangle + \oint_C \mathrm{d}\bar{z}\,\bar{\epsilon}(\bar{z})\,\langle\bar{T}(\bar{z})X\rangle. \tag{17}$$

Then, when inserted into any correlation function, the stress energy tensor satisfies the standard operator product expansion

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{18}$$

with central charge $c = -2$. Finally, the correlation functions of the field θ satisfy the third-order differential equation coming from the condition of degeneration of the operator $(1, 3)$ with weight $h_{1,3} = 0$ on the third level. This equation actually becomes of second order for the field $\partial\theta$ and, in its turn, coincides with the condition of degeneration of the operator $(2, 1)$ with weight $h_{2,1} = 1$ on the second level.

The twist field σ can be defined on the lattice by means of a construction similar to that for the disorder operator in the Ising model [14]. Namely, consider a point on the dual lattice (formed by the centres of faces of the original lattice) and draw some path on the dual lattice from this point to infinity. Change the weights of all the bonds intersected by the path. The end point of the dislocation line is called the disorder field or twist field. Its conformal properties are similar to those of the Grassmann field θ. Namely, its correlation functions are non-trivial even in the presence of zero mode. These correlation functions can be found from the condition of degeneration of the operator $(1, 2)$ with weight $h_{1,2} = -1/8$ on the second level [14]

$$\langle\sigma_1\sigma_2\rangle = \sqrt{r_{12}} \tag{19a}$$

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle = \pi\sqrt{r_{12}r_{34}}\sqrt{|\eta(1-\eta)|}\{F(\eta)\bar{F}(1-\bar{\eta}) + \bar{F}(\bar{\eta})F(1-\eta)\} \tag{19b}$$

where $F(\eta) = {}_2F_1(1/2, 1/2; 1; \eta)$ is a hypergeometric function and $\eta = (z_{13}z_{24})/(z_{12}z_{34})$.

Mixed four-point correlation functions of the fields θ and σ can also be found using standard techniques of CFT

$$\langle\theta_1^*\theta_2\sigma_3\sigma_4\rangle = 2\sqrt{r_{34}}\{H(\eta) + \bar{H}(\bar{\eta})\}. \tag{20}$$

Here $H(\eta) = \ln(\sqrt{\eta} + \sqrt{\eta - 1})$.

This means that both the Grassmann field θ and its twist field σ can be considered as primary conformal fields with weights $(h_\theta, \bar{h}_\theta) = (0, 0)$ and $(h_\sigma, \bar{h}_\sigma) = (-1/8, -1/8)$ provided that the vacuum state has been defined as having zero norm. These fields are unique in having both the property and non-positive conformal weights.

4. Krein space

We come now to the following question: what are the properties of the space of states wherein the operators of the theory act? This, obviously, cannot be standard Hilbert space since the vacuum vector of the theory has zero norm, $\langle 0|0\rangle = 0$. Instead, we should consider the Krein space with indefinite metric [15]. The Krein space $\mathcal{H}$ is defined as a Hilbert space with definite metric $(x|y)$ and additional involutive unitary operator of canonical symmetry J

$$J = J^{-1} = J^\dagger. \tag{21}$$

Using the operator of canonical symmetry one can introduce in the Hilbert space $\mathcal{H}$ the indefinite metric

$$\langle x|y\rangle = (Jx|y). \tag{22}$$

Thus, the Krein space admit canonical decomposition into the orthogonal sum of the two subspaces

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- = P^+\mathcal{H} \oplus P^-\mathcal{H} \tag{23}$$

where

$$P^+ = \frac{1 + J}{2} \qquad P^- = \frac{1 - J}{2} \tag{24}$$

are projectors to the states with positive and negative norms, respectively (in the indefinite metric).

If the spectrum of an operator A in the indefinite metric is real, $\mathrm{Im}\langle x|A|x\rangle = 0$, it is called J-Hermitean and satisfies the condition

$$JA^\dagger J = A. \tag{25}$$

To define the operator of canonical symmetry, let us consider mode expansion of the Grassmann field $\theta(z)$

$$\theta(z, \bar{z}) = \chi_0 + \theta_0 \ln|z|^2 - \sum_{n\neq 0}\left(\frac{\theta_n}{n}z^{-n} + \frac{\bar{\theta}_n}{n}\bar{z}^{-n}\right) \tag{26}$$

with the anticommutation relations [22, 23]

$$\{\chi_0^*, \theta_0\} = -\{\chi_0, \theta_0^*\} = 1 \qquad \{\theta_n^*, \theta_m\} = n\,\delta_{n+m}. \tag{27}$$

The only operator we can construct from zero modes of the Grassmann field that satisfies all the necessary conditions is

$$J = \chi_0^*\chi_0 + \theta_0\theta_0^* + (\chi_0^*\theta_0 - \chi_0\theta_0^*) - 2\chi_0^*\chi_0\theta_0\theta_0^*. \tag{28}$$

Indeed, this operator is involutive and unitary due to anticommutation relations in equation (27). This operator is also invariant under the group SU(1, 1) of the global gauge transformations of the action in (12).

Now, we can calculate the correlation functions of the Grassmann field both in the definite and in the indefinite metric. Let us first introduce the vacuum vector $|0)$ normalized in the definite metric, $(0|0) = 1$ and let us assume that $\theta_n|0) = 0$ for $n \geqslant 0$ and $(0|\chi_0 = 0$, $(0|\theta_n = 0$ for $n < 0$. In other words, in the definite metric we consider operators χ_0 and θ_n with $n < 0$ as creation operators and θ_n with $n \geqslant 0$ as annihilation operators. Then, straightforward calculation in the definite metric leads to the correlation functions that obey standard Wick rules with the two-point function

$$(0|\theta^*(z_1, \bar{z}_1)\theta(z_2, \bar{z}_2)|0) = -\ln|z_1 - z_2|^2. \tag{29}$$

Simple analysis of the transformation properties of this equation leads to the conclusion that the definite vacuum *is not* invariant under projective SL(2,C) transformations. This is due to the fact that the Grassmann field θ is scalar and does not change under projective transformations, whereas the right-hand side of the equation does change. Similar facts for the ghost (ξ, η)-system have already been noticed by Dixon *et al* [24].

The same calculation in the indefinite metric with the right vacuum vector $|0\rangle = |0)$ orthogonal to the left vacuum vector $\langle 0| = (J|$ so that $\langle 0|0\rangle = (J|0) = 0$ leads to the correlation functions of (14). Hence, the indefinite vacuum is invariant under projective transformations.

To understand this better let us consider the Lourant expansion of the stress-energy tensor

$$T(z) = \sum_n L_n z^{-n-2}. \tag{30}$$

The Virasoro generators of conformal transformations can be written in terms of modes of the Grassmann field as

$$L_n = \sum_k \theta^*_{n-k}\theta_k \tag{31}$$

$$L_0 = 2\theta_0\theta_0^* + \sum_{k=1}^{\infty}(\theta^*_{-k}\theta_k - \theta_{-k}\theta^*_k). \tag{32}$$

The normal ordering in the definition of the operator L_0 which generate the uniform dilation transformation, $\tilde{z} = \mathrm{e}^{\epsilon} z$, can be found from the transformation properties of the modes of the scalar Grassmann field

$$\tilde{\chi}_0 = \chi_0 - 2\epsilon\theta_0 \qquad \tilde{\theta}_0 = \theta_0 \qquad \tilde{\theta}_n = \mathrm{e}^{n\epsilon}\theta_n. \tag{33}$$

More generally, an operator A changes under such a transformation as

$$\tilde{A} = \mathrm{e}^{\epsilon L_0} A\, \mathrm{e}^{-\epsilon L_0}. \tag{34}$$

Note, that the generator of dilation transformations L_0 is J-Hermitean and, hence, all the critical exponents of the theory are real.

Finally, let us note that although the right vacuum $|0)$ does not change under the dilation transformation, the left vacuum does change

$$|\tilde{0}) = \mathrm{e}^{\epsilon L_0}|0) = |0) \tag{35a}$$

$$(\tilde{0}| = (0|\mathrm{e}^{-\epsilon L_0} = (0| - 2\epsilon(J|. \tag{35b}$$

In contrast, both indefinite vacua are SL(2,C) invariant.

As an example of the field with positive conformal weight let us consider correlation functions of the local energy operator

$$\varepsilon_0 =: \partial_\mu\theta^*\partial^\mu\theta :=: \lim_{1\to 0}\{\partial_\mu\theta_0^*\partial^\mu\theta_1 - 4\pi\delta(r_{01})\}. \tag{36}$$

This is primary with conformal weight $(h_\varepsilon, \bar{h}_\varepsilon) = (1, 1)$. Its correlation functions can be found from equations (14) and are all trivial, $\langle \varepsilon_1 \dots \varepsilon_N \rangle = 0$, when calculated in the indefinite metric. However, in the definite metric we have

$$(0|\varepsilon_1\varepsilon_2|0) = -\frac{8}{(r_{12})^4} \tag{37a}$$

$$(0|\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4|0) = \frac{64}{(r_{12}r_{34})^4} + \frac{64}{(r_{13}r_{24})^4} + \frac{64}{(r_{14}r_{23})^4}. \tag{37b}$$

This property is common for all primary fields with positive conformal weights. Their correlation functions have to be calculated and are non-zero only in the definite metric.

We conclude that the physical meaning of the operator of canonical symmetry J is to subtract or add the zero mode of the Laplacian operator.

5. Dirichlet operator and Green function

There is another way to supress the zero mode of the Laplacian operator. Let us consider a conducting plane with a current $I = 1$ entering the plane at a point $\boldsymbol{r}_1$ and leaving it at a point $\boldsymbol{r}_2$. The voltage difference between sites $\boldsymbol{r}_3$ and $\boldsymbol{r}_4$ on the plane is given by the four-point function $\langle \theta_1^*\theta_2^*\theta_3\theta_4 \rangle$. If the plane is grounded at the point $\boldsymbol{r}_0$ and a current $I = 1$ enters the plane at a point $\boldsymbol{r}_1$ (and leaves it at the grounded point $\boldsymbol{r}_0$) then the voltage at a site $\boldsymbol{r}_3$ is given by the Green function $G_0(\boldsymbol{r}_1, \boldsymbol{r}_3)$.

The operator $\mathcal{D}_0$ that corresponds to the grounded point $\boldsymbol{r}_0$ can, obviously, be considered as the product of the field θ_0 with its complex conjugate θ_0^* at the same point. We will call it the *Dirichlet operator* since it imposes the Dirichlet boundary conditions on the Grassmann field. From the four-point correlation functions in equations (14*c*), (19*b*) and (20) we conclude that the Dirichlet operator can be defined through either of the following two limits

$$\mathcal{D}_0 = \lim_{1\to 0} \left\{ \theta_0^* \theta_1 \right\} = \lim_{1\to 0} \left\{ \frac{\sigma_0\sigma_1}{\sqrt{r_{01}}} \right\}. \tag{38}$$

With the help of this operator the Green function with the Dirichlet boundary conditions at the point $\boldsymbol{r}_0$ can be represented as

$$G_0(\boldsymbol{r}_1, \boldsymbol{r}_3) = \lim_{2,4\to 0} \langle \theta_1^*\theta_2^*\theta_3\theta_4 \rangle = -\langle \mathcal{D}_0 \theta_1^* \theta_3 \rangle. \tag{39}$$

We have to be careful merging different points of the four-point functions since they diverge logarithmically in the limit. These divergences are always present in the Green function in the thermodynamic or continuous limit [19–21]. To treat them carefully we should introduce the lower cut off $\sim a$ (lattice spacing) and should replace the vanishing distance between different merging points by the square root from the metric on the plane at the same point: $\lim_{1\to 0} r_{01} \sim (g(\boldsymbol{r}_0)a^2)^{1/2}$. Thus, for the Green function we obtain

$$G_0(\boldsymbol{r}_1, \boldsymbol{r}_3) = \ln \frac{(r_{10})^2 (r_{30})^2}{(r_{13})^2 g(\boldsymbol{r}_0)} \tag{40}$$

where the factor $\sim a^2$ is absorbed into the metric. It is well known and can be verified directly from this equation that the Green function of the Laplacian operator with Dirichlet boundary conditions is invariant under projective conformal transformations (see e.g. [21]). This is quite natural since the Dirichlet operator has been defined as the product of scalar fields θ and θ^*. The correlation functions of the Dirichlet operator can be found from the functions defined in equations (14), (19) and (20) and are all projectively invariant

$$\langle \mathcal{D}_1 \rangle = 1 \qquad \langle \mathcal{D}_1 \mathcal{D}_2 \rangle = \ln \frac{g_1 g_2}{(r_{12})^4} \qquad \dots . \tag{41}$$

Since the correlation functions of the Dirichlet operator involve metric fields, the Dirichlet operator is, obviously, not conformal.

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